5. APPLICATION TO THE PROOF OF INVARIANCE OF THE INTERVAL. Suppose the coordinates $\mathbf{x} = (t, x, y, z)$ in K and $\mathbf{x}' = (t', x', y', z')$ in K' are connected by a linear transformation, so $\mathbf{x}' = L\mathbf{x}$ for some 4×4 matrix L. Let $q(\mathbf{x}) = -c^2t^2 + x^2 + y^2 + z^2 = \mathbf{x}^t Q\mathbf{x}$, where Q is the diagonal matrix with diagonal entries $(-c^2, 1, 1, 1)$. Let $r(\mathbf{x}) = -c^2t'^2 + x'^2 + y'^2 + z'^2 = (L\mathbf{x})^t QL\mathbf{x} = \mathbf{x}^t(L^t QL)\mathbf{x}$, so $r(\mathbf{x}) = \mathbf{x}^t R\mathbf{x}$, where $R = L^t QL$. Now q is indefinite, and $r(\mathbf{x}) = 0$ precisely when $q(\mathbf{x}) = 0$, from $(*)$ above. So the conditions of Theorem 1 are in force, and we may conclude that r is proportional to q, which is equivalent to the statement from $(*)$ that we wanted to prove, namely that s'^2 is proportional to s^2.

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Monotone Convergence Theorem for the Riemann Integral

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Abstract. The monotone convergence theorem holds for the Riemann integral, provided (of course) it is assumed that the limit function is Riemann integrable. It might be thought, though, that this would be difficult to prove and inappropriate for an undergraduate course. In fact the identity is elementary: in the Lebesgue theory it is only the integrability of the limit function that is deep. This article shows how to prove the monotone convergence theorem for Riemann integrals using a simple compactness argument (i.e., invoking Cousin's lemma). This material could reasonably and appropriately be used in classroom presentations where the students are indoctrinated on this antiquated, but still popular, integration theory.

The monotone convergence theorem is usually stated and proved for the Lebesgue integral, but there is little difficulty in formulating and proving a version for the Riemann integral.

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Monotone Convergence Theorem. *Let $\{f_n\}$ be a nondecreasing sequence of Riemann integrable functions on the interval $[a, b]$. Suppose that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every x in $[a, b]$. Then, provided f is also Riemann integrable on $[a, b]$,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx. \tag{1}$$

This theorem should have been useful in many calculus presentations, but it does not appear in any of the usual textbooks. Perhaps the reason is that, because the Lebesgue version of the theorem is deep, it might follow that this version too is at some deeper level than the students should be taken. But it is not the identity (1) that is deep in Lebesgue's theory, but his conclusion that such a function must be integrable. Here we are assuming integrability so the theorem is entirely elementary.

Teaching this theorem offers the instructor some real opportunities. First is the chance to introduce a major theorem of integration theory at an elementary level and discuss its importance and how it must be improved. Second is the occasion (always tempting) to launch a polemic against the Riemann integral. The unfortunate hypothesis that the limit function is integrable is essential here, but reduces the theorem to a curiosity: in most applications we would know nothing more about the limit function than that it is a pointwise limit of integrable functions and would have serious difficulty finding some property that would assure Riemann integrability.

The proof is nothing but some manipulations of Riemann sums and surely as accessible as any of the other theorems proved in Riemann integration theory.

We need a few preliminaries. By a *partition* of an interval $[a, b]$ we mean a collection

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, \dots, n\}$$

of interval-point pairs for which each $w_i \in [u_i, v_i]$ and the intervals form a collection of nonoverlapping intervals whose union is $[a, b]$. Any subset of a partition is a *subpartition*. The use of the Greek letter π to denote a partition will, no doubt, distress a calculus class but I am addicted to it.

The Riemann integral, defined as a limit of Riemann sums, possesses also this apparently stronger property:

($\star$) *If the function f is integrable in the Riemann sense on an interval $[a, b]$ then, for every $\epsilon > 0$, there is a $\delta > 0$ so that*

$$\sum_{([u,v],w)\in\pi} \left| \int_u^v f(x)\,dx - f(w)(v-u) \right| < \epsilon \tag{2}$$

whenever π is a partition or subpartition of the interval $[a, b]$ such that $v - u < \delta$ for every pair $([u, v], w) \in \pi$.

This well-known property is seldom proved in calculus courses although it is only a simple computation using Riemann sums. It should be proved in any case since, from (2), one immediately deduces

$$\sum_{([u,v],w)\in\pi} \omega_f([u, v])(v-u) < 2\epsilon \tag{3}$$

which is Riemann's famous characterization of integrability expressed in terms of the oscillation ω_f of the function f on subintervals. That, in turn, allowed Lebesgue to easily formulate his more famous characterization. Thus there is a lot of narrative potential in ($\star$).

While the proof of ($\star$) is elementary it is sufficiently subtle that a student might have trouble attempting it without coaching. The method in [**2**, p. 77] works here and can be attributed to Saks [**3**] who used it in his study of the Burkill integral.

Now we may present the proof of the monotone convergence theorem. For each integer n, let $g_n = f - f_n$. The sequence of integrable functions $\{g_n\}$ is nonnegative and monotone decreasing with $\lim_{n\to\infty} g_n(x) = 0$ at each x.

Let $\epsilon > 0$ and write $\eta = \epsilon/(b - a + 1)$. For each integer n, use ($\star$) to choose a positive number δ_n so that

$$\sum_{([u,v],w)\in\pi} \left| \int_u^v g_n(x)\,dx - g_n(w)(v-u) \right| < \eta 2^{-n}$$

whenever π is a partition of the interval $[a, b]$ such that $v - u < \delta_n$ for every pair $([u, v], w) \in \pi$. Choose, for each $x \in [a, b]$, the first integer $N(x)$ so that

$$g_n(x) < \eta \text{ for all integers } n \geq N(x)$$

and, for $j = 1, 2, 3, \dots$, let

$$E_j = \{x \in [a, b] : N(x) = j\}.$$

We use these sets to define $\delta(x) = \delta_j$ whenever x belongs to the corresponding set E_j.

Take any partition π of the interval $[a, b]$ for which $v - u < \delta(w)$ for every pair $([u, v], w) \in \pi$. That such partitions exist is the conclusion of Cousin's lemma. That lemma plays the same role as, and is equivalent to, the nested interval property on the real line. Many of the theorems of the calculus can conveniently use either argument. (See the discussions in [**1**], [**4**], and [**5**].)

Let N be the largest value of $N(w)$ for the finite collection of pairs $([u, v], w)$ in π. We carve the partition π into a finite number of disjoint subsets by writing

$$\pi_j = \{([u, v], w) \in \pi : w \in E_j\}$$

for integers $j = 1, 2, 3, \dots, N$. Note that $\pi = \pi_1 \cup \pi_2 \cup \cdots \cup \pi_N$ and that these collections are pairwise disjoint.

Now let m be any integer greater than N. We compute

$$\begin{aligned}
0 \leq \int_a^b g_m(x)\,dx &= \sum_{([u,v],w)\in\pi} \left(\int_u^v g_m(x)\,dx \right) \\
&= \sum_{j=1}^{N} \left(\sum_{([u,v],w)\in\pi_j} \left(\int_u^v g_m(x)\,dx \right) \right) \leq \sum_{j=1}^{N} \left(\sum_{([u,v],w)\in\pi_j} \left(\int_u^v g_j(x)\,dx \right) \right) \\
&\leq \sum_{j=1}^{N} \left[\sum_{([u,v],w)\in\pi_j} g_j(w)(v-u) + \eta 2^{-j} \right]
\end{aligned}$$

$$< \sum_{j=1}^{N} \left[\sum_{([u,v],w)\in\pi_j} \eta(v-u) + \eta 2^{-j} \right] < \eta(b-a+1) = \epsilon.$$

The identity

$$\int_a^b f(x)\,dx - \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0$$

follows.

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A Proof of a Version of a Theorem of Hartogs

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Abstract. It is proved that a formal power series in s complex variables is convergent, if it is convergent on each line through the origin.

A formal power series in s variables

$$f = \sum_{i_1,\dots,i_s\geq 0} a_{i_1,\dots,i_s} z_1^{i_1}\cdots z_s^{i_s}$$

with complex coefficients $a_{i_1,\dots,i_s} \in \mathbb{C}$ is called *convergent* if is absolutely convergent in a neighbourhood of 0. This means that there exists a positive real number r such that

$$f = \sum_{i_1,\dots,i_s\geq 0} |a_{i_1,\dots,i_s}| r^{i_1+\cdots+i_s} = \sum_{n=0}^{+\infty} \left(\sum_{i_1+\cdots+i_s=n} |a_{i_1,\dots,i_s}| \right) r^n < +\infty.$$

The aim of this note is to prove that the convergence of a formal power series can be established by checking convergence only on the lines passing through the origin of $\mathbb{C}^s$.

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